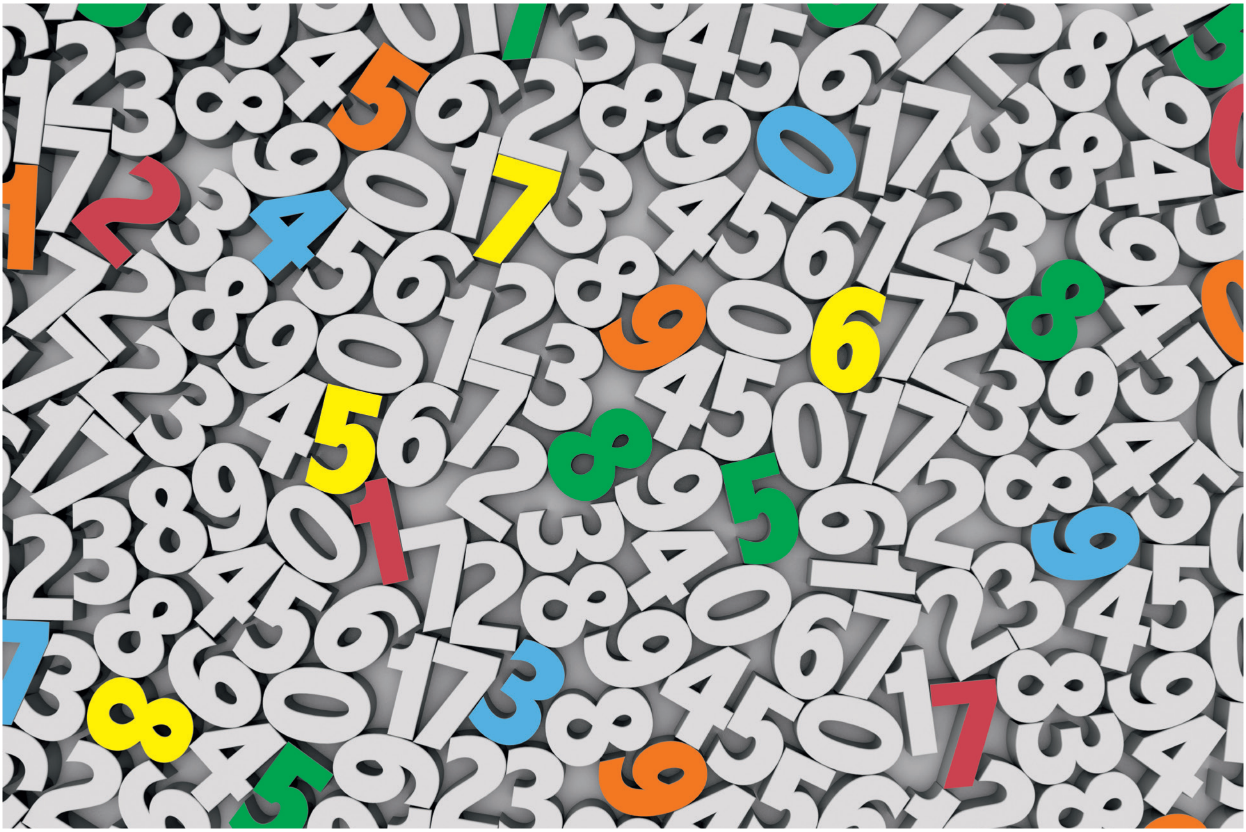

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INTRODUCTION TO LINEAR ALGEBRA

Casa Editrice Ambrosiana

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Contents

Preface

This is the English translation of the textbook “Introduzione all’algebra lineare”, originally published to cater this material to engineering and computer science students, but later on adopted as textbook by other majors, like Genomics, whose teaching activities are proposed entirely in English.

Though we propose the full proof of all of our statements, we introduce the subject with a lot of examples and intuitive explanations to guide the students through this beautiful subject.

We would like to thank the Department of Mathematics, which supported us through these years of teaching and also our many students, who have encouraged us through this journey and alerted us about the typos of the previous version: if this book is improved it is also because of their contribution.

Finally our last thank to our families, whose encouragement and support has made this book possible.

1 Introduction to linear systems

We want to solve linear systems with real coefficients using a method known as *Gauss algorithm*. Later on, we will also use this method to solve other problems and, at the same time, we will interpret linear systems as special cases of a much deeper theory.

1.1 LINEAR SYSTEMS: FIRST EXAMPLES

A *linear equation* is an equation where the unknowns appear with degree 1, that is an equation of the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \tag{1.1}$$

where $a_1, a_2, \ldots, a_n$ and b are assigned numbers and $x_1, x_2, \ldots, x_n$ are the unknowns. The numbers $a_1, \ldots, a_n$ are called *coefficients* of the linear equation, b is called *known term.* If $b = 0$ the equation is said to be *homogeneous.* A *solution* of the equation (1.1) is a n-tuple of numbers $(s_1, s_2, \ldots, s_n)$, that gives an equality when put in place of the unknowns. For example $(3, -1, 4)$ is a solution of the equation $2x_1 + 7x_2 - x_3 = -5$ because $2 \cdot 3 + 7 \cdot (-1) - 4 = -5$.

A *linear system of m equations in n unknowns* $x_1, x_2, \ldots, x_n$ is a set of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ that must be simultaneously satisfied:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1.2}$$

The numbers $a_{11}, \ldots, a_{1n}, \ldots, a_{m1}, \ldots, a_{mn}$ are called the system coefficients, while $b_1, \ldots, b_m$ are called the known terms. If $b_i = 0$ for every $i = 1, \ldots, m$, the system is said to be *homogenous.* A *solution* of the linear system (1.2) is a n-tuple $(s_1, s_2, \ldots, s_n)$ of numbers that satisfies all the system equations. For example $(1, 2)$ is a solution of the linear system

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 - x_2 = -1 \end{cases}$$

In this book we will deal exclusively with linear systems with *real coefficients* that is, systems of the form (1.2) in which all the coefficients a_{ij} of the unknowns and all known terms b_i are real numbers. The solutions that we will find, therefore, will always be ordered n-tuples of real numbers.

Given a linear system, we aim at answering the following questions:

1. Does the system admit solutions?

2. If so, how many solutions does it admit and what are they?

In certain cases it is particularly easy to answer these questions. Let us see some examples.

Example 1.1.1

Consider the following linear system in the unknowns x_1, x_2:

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 + x_2 = 1 \end{cases}$$

It is immediate to observe that the sum of two real numbers cannot be simultaneously equal to 3 and 1. Thus, the system does not admit solutions. In other words, when the conditions assigned by the two equations of the system are incompatible, then the system does not have solutions.

The example above justifies the following definition:

Definition 1.1.2 A system is said *compatible* if it admits solutions.

Examples 1.1.3

Consider the following linear system in the unknowns x_1, x_2:

$$\begin{cases} x_1 + x_2 = 3 \\ x_2 = -1 \end{cases}$$

Substituting in the first equation the value of x_2 obtained from the second one, we get: $x_1 = 3 - x_2 = 3 + 1 = 4$. The system is therefore compatible and admits a unique solution: $(4, -1)$. In this example two variables are assigned (the unknowns x_1 and x_2) and two conditions are given (the two equations of the system). These conditions are compatible, that is they are not contradictory, and are "independent" meaning that they cannot be obtained one from the other. In summary:

two real variables along with two compatible conditions give one and only one solution.

Example 1.1.4

Now consider the linear system in the unknowns x_1, x_2.

$$\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 6 \end{cases}$$

Unlike what happened in the previous example, here the conditions given by the two equations are not "independent", in the sense that the second equation is obtained by multiplying the first by 2. The two equations give the same relation between the variables x_1 and x_2. Then, solving the linear system means simply solving the equation $x_1+x_2=3$. This equation certainly has solutions: for example, we saw in the previous example that $(4,-1)$ is a solution, but also $(1,2)$ or $(0,3)$ are solutions. Then, exactly, how many the solutions are there? And how can we find them out? In this case, we have two variables and one condition on them. This means that a variable is free to vary in the set of real numbers, which are infinitely many. The equation allows us to express a variable, say x_2, as a function of the other variable x_1. The solutions are all expressible in the form: $(x_1, 3-x_1)$. With this, we mean that the variable x_1 can take all the infinite real values, and that in order for the equation $x_1+x_2=3$ to be satisfied, it must be $x_2=3-x_1$. A more explicit way, but obviously equivalent, to describe the solutions, is $\{(t,3-t)|t\in\mathbb{R}\}$. Of course, we could decide to vary the variable x_2 and express x_1 as a function of x_2. In that case we would give the solutions in the form $(3-x_2, x_2)$, or equivalently we say that the set of solutions is: $\{(3-s,s)|s\in\mathbb{R}\}$. In summary:

two real variables along with one condition give infinitely many solutions.

Definition 1.1.5 Two linear systems are called *equivalent* if they have the same solutions.

In Example 1.1.4 we observed that the linear system

$$\begin{cases} x_1+x_2=3 \\ 2x_1+2x_2=6 \end{cases}$$

is equivalent to the equation $x_1+x_2=3$. Of course, being able to understand if two systems are equivalent can be very useful; for example, we can try to solve a linear system by reducing it to an equivalent one, but easier to solve.

In the next section we will introduce some useful concepts to simplify the way we write a linear system.

1.2 MATRICES

Given two natural numbers m, n, a $m\times n$ *matrix* with real coefficients is a table of mn real numbers placed on m rows and n columns. For example:

$$\begin{pmatrix} 5 & -6 & 0 \\ 4 & 3 & -1 \end{pmatrix}$$

is a 2×3 matrix.

If $m=n$ the matrix is said to be *square* of order n. For example

$$\begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 3 \end{pmatrix}$$

is a square matrix of order 2.

We denote by $\mathrm{M}_{m,n}(\mathbb{R})$ the set of $m\times n$ matrices with real coefficients and simply by $\mathrm{M}_n(\mathbb{R})$ the set of square matrices of order n with real coefficients.

Given a matrix A, the number that appears in the i-th row and j-th column of A is called the (i, j) entry of A.

For example in the matrix

$$A = \begin{pmatrix} 5 & -6 & 0 \\ 4 & 3 & -1 \end{pmatrix}$$

the $(1, 3)$ entry is 0, while the $(2, 2)$ entry is 3. Of course, two $m \times n$ matrices A and B are equal if their entries coincide, that is, if the (i, j) entry of A coincides with (i, j) entry of B, for every $i = 1, \ldots, m$ and for every $j = 1, \ldots, n$.

Given a generic $m \times n$ matrix, we can write it synthetically as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

where a_{ij} is the (i, j) entry, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

We now want to define the product rows by columns between two matrices A and B, in the case where the rows of A have the same length as the columns of B.

If A is a $m \times s$ matrix and B is a $s \times n$ matrix, we define the product c_{ij} of the i-th row of A and j-th column of B in the following way:

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{is} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{is}b_{sj}$$

which we also write as:

$$c_{ij} = \sum_{h=1}^{s} a_{ih} b_{hj}$$

In practice, we have multiplied, in order, the coefficients of the i-th row of A by the coefficients of the j-th column of B, then we have added the numbers obtained.

For example if we have

$$A = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & -2 & 2 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ -3 & 5 \\ 1 & 0 \\ 2 & -1 \end{pmatrix}$$

then

$$c_{12} = 1 \cdot 1 + 0 \cdot 5 + 3 \cdot 0 + (-1) \cdot (-1) = 2$$
$$c_{31} = 1 \cdot 0 + 0 \cdot (-3) + (-1) \cdot 1 + 0 \cdot 2 = -1$$

At this point we define the product of A and B as

$$C = AB = (c_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} .$$

The matrix C is the product of A and B and it is a $m \times n$ matrix.

In the previous example we have that

$$C = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & -2 & 2 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -3 & 5 \\ 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 10 & -11 \\ -1 & 1 \end{pmatrix}$$

We note that, in general, the number of rows of AB is equal to the number of rows of A and the number of columns of AB is equal to the number of columns of B.

We also observe that the product of a $m \times n$ matrix and a $n \times 1$ matrix (*i.e.* a vector in $\mathbb{R}^n$) results in a $m \times 1$ matrix, that is, a vector in $\mathbb{R}^m$.

Proposition 1.2.1 *The product operation between matrices enjoys the following properties:*

1. *associative, that is,* $(AB)C = A(BC)$ *where* A, B, C *are matrices such that the products that appear in the formula are defined;*
2. *distributive, that is,* $A(B + C) = AB + AC$, *provided that the sum and product operations that appear in the formula are defined.*

Proof – The proof is a calculation and amounts to applying the definition. We show only the associativity of the product. Consider $A \in \mathrm{M}_{m,s}(\mathbb{R})$, $B \in \mathrm{M}_{s,r}(\mathbb{R})$, $C \in \mathrm{M}_{r,n}(\mathbb{R})$. We observe that:

$$(AB)_{iu} = \sum_{h=1}^{s} a_{ih} b_{hu}, \ \ (BC)_{hj} = \sum_{u=1}^{r} b_{hu} c_{uj}$$

then

$$\begin{aligned} ((AB)C)_{ij} &= \sum_{u=1}^{r} (AB)_{iu} c_{uj} = \sum_{u=1}^{r} (\sum_{h=1}^{s} a_{ih} b_{hu}) c_{uj} \\ &= \sum_{u=1}^{r} \sum_{h=1}^{s} a_{ih} b_{hu} c_{uj} = \sum_{h=1}^{s} \sum_{u=1}^{r} a_{ih} b_{hu} c_{uj} \\ &= \sum_{h=1}^{s} a_{ih} \left(\sum_{u=1}^{r} b_{hu} c_{uj} \right) = \sum_{h=1}^{s} a_{ih} (BC)_{hj} = (A(BC))_{ij} \end{aligned}$$

The proof of distributivity is similar. □

Note that the product operation between matrices is not commutative. Even if the product AB between two matrices A and B is defined, the product BA could not be defined. For example if

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

we have that

$$AB = \begin{pmatrix} 1 & -1 \\ 2 & -1 \\ -1 & 1 \end{pmatrix}$$

while BA is not defined. Similarly if

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$$

we have that

$$AB = \begin{pmatrix} -1 & 5 \\ 0 & -6 \end{pmatrix} \qquad BA = \begin{pmatrix} -1 & -5 \\ 0 & -6 \end{pmatrix}$$

1.3 MATRICES AND LINEAR SYSTEMS

Let us now see how it is possible to use matrices and the product rows by columns to describe a linear system.

Consider a linear system of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases}$$

We can write this system in matrix form as follows:

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

and then using the product rows by columns in the following way:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or, more synthetically, as

$$A\underline{x} = \underline{b},$$

where $A = (a_{ij})$ is the $m \times n$ matrix which has as entries the coefficients of the unknowns,

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

A Solutions of some suggested exercises

Chapter 1: Introduction to linear systems

1.6.1 a) $x = y = 0$, $z = 1$.
b) The system has no solutions.

1.6.3 The system has a unique solution for $k \neq 0, 1$. For $k = 0$, the system has infinitely many solutions depending on one parameter, for $k = 1$ the system does not have solutions.

Chapter 2: Vector Spaces

2.6.1 a), b), d), e), f), l) are subspacess. c), g), h), i), m) are not subspaces.

2.6.4 X is not a vector subspace.

Chapter 3: Linear Combinations and linear indipendence

3.4.1 a), d), e) are linearly independent sets. b), c) are linearly dependent sets.

3.4.3 Yes.

3.4.4 $k = \pm\sqrt{3}$.

3.4.5 $k \neq -1/2$.

3.4.6 $k = -2/5$ and $k = 0$.

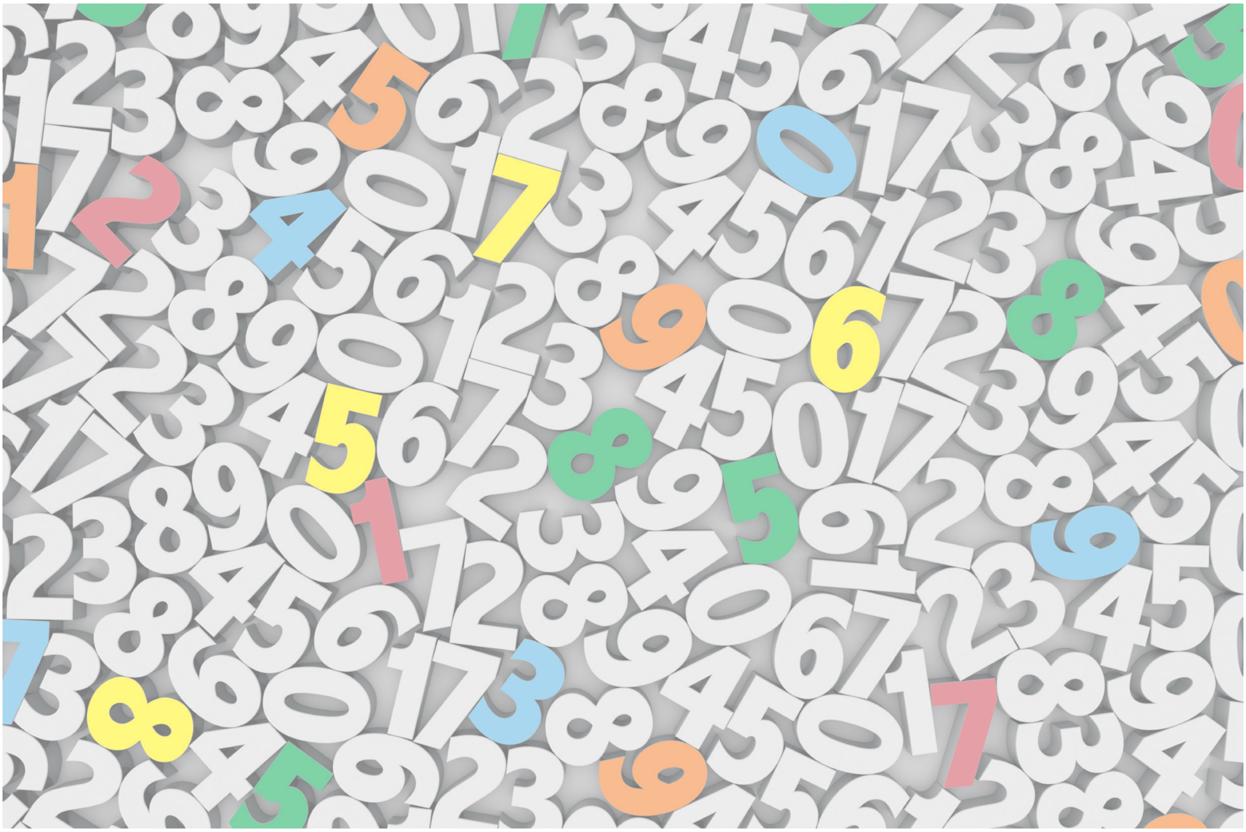
3.4.9 a) $k \neq 2, -1$, b) $k \neq 2$.

3.4.10 a) The system has a unique solution for $k \neq 0, 1$, it has no solutions for $k = 0, 1$. b) $k \neq 0, 1$.

3.4.12 The given vectors are always linearly dependent.

3.4.13 a) $k \neq 0, 5/3$. For $k = 0$ we have $\mathbf{v}_2 = \mathbf{v}_3$.

3.4.14 a) $k \neq 0, -2$. b) $k \neq 0, -2$.



Linear Algebra gives the essential mathematical tools to face real-world problems by converting them into linear models: problems then become easy to solve. There are no «Linear Algebras» but a subject with many different flavours, depending on the realm of application.

This textbook is suitable especially for Computer Science and Engineering students, as it offers a practical yet rigorous approach to solve various types of problems and their applications.

Together with the proofs, the authors provide many examples and explanations of each result.

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FIORESI*INTR LINEAR ALGEBRA (CEA
ISBN 978-88-08-92029-4
9 788808 920294
0 1 2 3 4 5 6 7 8 (64B)

Al pubblico € 23,00 • • •
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